J. Fluid Mech. (1968), *vol.* 34, *part* 2, *pp.* 263–272
Printed in Great Britain

A problem of gravity wave drag at an interface

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(Received 26 October 1967 and in revised form 19 March 1968)

A thin disk, immersed in a fluid consisting of two superposed homogeneous inviscid liquids, moves in its own vertical plane. A Green's function is found for the potential and a formula for the gravity wave energy is found and evaluated for a specific case.

1. Introduction

When a body traverses a stratified fluid under gravity internal waves are set up. In the particular case of two superposed homogeneous liquids of different densities these waves take the form of simple interfacial waves whose amplitude decreases away from the level of density discontinuity. There arises the problem of determining the energy which is propagated outwards along the interface. If the body is thin and its central plane always coincides with a fixed vertical plane, a familiar linearization enables estimates to be found in a comparatively simple manner, provided the angular velocity of the body is always zero. Hudimac (1961) has considered the problem of wave drag to horizontal motion on the surface of a stratified ocean, and Crapper (1967) has discussed the associated wave formation. Warren (1961) studied the case of strictly vertical motion of a thin body. Here the purpose is an investigation of the case of general motion in a vertical plane, excluding the effects of angular velocity. Thus the position of a given point of the body, hereafter called its centre, is given as a function of the time, while longitudinal and transverse axes fixed in the body always maintain the same angle of elevation with the horizontal. Moreover, the motion of the centre is confined to the fixed vertical plane which always contains the central plane of symmetry of the body. The thickness of the body varies from point to point, but in the strict mathematical sense its maximum thickness is infinitesimal compared with its lateral dimensions, and so the body may be thought of as a thin disk which is free to move in its own vertical plane. The basic method is given by Warren (1961, 1968), and a Green's function is found. A drag formula and an expression for the gravity wave energy follow. An important case is one in which the disk moves with a fixed speed on a steady course in a straight line. The formula for the wave energy is evaluated in this instance for a specific shape of the disk and a small value of the density difference between the upper and lower fluids. The dependence of this energy upon the angle of incidence (γ) and the angle of inclination (β) is shown graphically. There is also the problem of estimating the accompanying lift and torque, but considerations of computer time and cost were discouraging and no attempt to find these associated quantities was made. However, in many cases the value of the drag is an indication of the magnitude and importance of these.

2. A Green's function for the potential

The submerged disk moves in its fixed vertical (y, z)-plane so that its angular velocity in this plane is zero, but otherwise the motion is arbitrary. The position of its centre at time t is

$$(0, y_0(t), z_0(t)). \tag{2.1}$$

The z-axis is positive upwards and the (x, y)-plane is defined by the undisturbed horizontal interface. The thickness of the disk varies from point to point so that its lateral surfaces are given by

$$x = \pm a\xi(y, z, t) \tag{2.2}$$

$$= \pm a\xi_0(y - y_0(t), z - z_0(t)), \tag{2.3}$$

where $2a$ is the maximum thickness. At points of the (y, z)-plane exterior to the disk, $\xi(y, z, t)$ is defined to be zero. By assumption the disk is thin so that if A is the mean height of the disk and B is the mean breadth, then $a \ll A$, $a \ll B$, and a small thickness parameter $\alpha \ll 1$ is defined by

$$\alpha = 2a/A. \tag{2.4}$$

The disk is originally at rest at and before time $t = t_0$ say, and so

$$\dot{y}_0(t) = \dot{z}_0(t) = 0 \quad \text{if} \quad t \leqslant t_0. \tag{2.5}$$

Here and hereafter a dot denotes a time derivative. In the absence of viscosity the velocity of a fluid particle may be expressed in the form $\nabla\phi$ where $\phi(x, y, z, t)$ is the velocity potential for both the upper and lower fluids. The boundary condition at the surface of the disk then is given approximately by

$$\partial\phi/\partial x|_{x=\pm 0} = \pm a\dot{\xi} \tag{2.6}$$

with an error $O(\alpha^2)$. This is the familiar thin body approximation to the boundary condition at the hull. Again for small α the wave amplitude at the interface is small and so the kinematic condition here is that approximately

$$\partial\phi/\partial z \text{ is continuous at } z = 0. \tag{2.7}$$

This is an infinitesimal wave approximation. Another condition on the potential is that

$$\phi \text{ is continuous at } x = 0. \tag{2.8}$$

The fluid is supposed to extend to infinity in all directions, and if $\bar{\rho}$ is the mean density of the upper and lower fluids then the current density ρ may be written as a function of the height z in the form

$$\rho = \rho(z) = \bar{\rho}(1 - \Delta\,\mathrm{sgn}\,z), \tag{2.9}$$

where $$0 \leqslant \Delta \leqslant 1.$$

Following the method outlined in Warren (1961) (see also Warren 1968), the strength G of the vortex sheet at the interface is introduced:

$$G(x,y,t) = \tfrac{1}{2}(\phi_{z=+0}-\phi_{z=-0}). \tag{2.10}$$

Continuity of pressure at the interface is expressed by continuity of

$$\rho(\partial^2\phi/\partial t^2 + g\,\partial\phi/\partial z)$$

at $z = 0$, with an error $O(\alpha^2)$ (see, for example, Lamb 1932). In terms of G this condition may be expressed as

$$\partial^2 G/\partial t^2 = \Delta g(\partial\phi/\partial z)_{z=0} + \tfrac{1}{2}\Delta\frac{\partial^2}{\partial t^2}(\phi_{z=+0}+\phi_{z=-0}). \tag{2.11}$$

The boundary condition at infinity is

$$\lim_{|\mathbf{r}|\to\infty}\phi(\mathbf{r},t) = \lim_{|\mathbf{r}|\to\infty}|\nabla\phi(\mathbf{r},t)| = 0, \tag{2.12}$$

for all finite t, and the initial conditions are

$$\phi = \dot{\phi} = G = \dot{G} = 0 \quad \text{at} \quad t = t_0. \tag{2.13}$$

Now if $\mathscr{T}(x)$ is a member of a suitable class of test functions then with the aid of (2.7), (2.8) and (2.12), equations (2.6) and (2.10) may be written

$$\int_{-\epsilon}^{\epsilon} dx\{\partial^2\phi/\partial x^2 - 2a\dot{\xi}\delta(x)\}\mathscr{T}(x) = O(\epsilon)$$

and

$$\int_{-\epsilon}^{\epsilon} dz\{\partial^2\phi/\partial z^2 - 2G\delta'(z)\}\mathscr{T}(z) = O(\epsilon),$$

where here and hereafter a prime denotes differentiation with respect to z. Here the potential is in its generalized form so that its differentiation at $x = 0$ and $z = 0$ is meaningful. Elsewhere the potential is harmonic, and so the field equation for the generalized potential reads

$$\nabla^2\phi = 2a\dot{\xi}\delta(x) + 2G\delta'(z). \tag{2.14}$$

Under a Fourier transform

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \exp\{-i(kx+ly+mz)\}$$

this equation becomes†

$$(k^2+l^2+m^2)\phi(k,l,m,t) + 2a\dot{\xi}(l,m,t) + 2imG(k,l,t) = 0, \tag{2.15}$$

where the functions now take their appropriate transformation meaning: for example

$$\xi(l,m,t) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\,\xi(y,z,t)\exp\{-i(ly+mz)\} = \exp\{-i\mathbf{K}.\mathbf{r}_0(t)\}\,\xi_0(l,m) \tag{2.16}$$

and

$$\xi_0(l,m) = \int_{-\infty}^{\infty} dY \int_{-\infty}^{\infty} dZ\,\xi_0(Y,Z)\exp\{-i(lY+mZ)\},$$

† The analysis here and immediately following is very similar to that given in Warren (1968).

where
$$\mathbf{K} = (k, l, m).$$
Then from (2.15) an application of the inverse operator
$$(1/2\pi)\int_{-\infty}^{\infty} dm\, e^{imz}$$
gives
$$\phi(k,l,z,t) = G(k,l,t)\, e^{-\mathscr{K}|z|}\operatorname{sgn} z - (a/\pi)\int_{-\infty}^{\infty} dm\, e^{imz}\,\dot{\xi}(l,m,t)/(\mathscr{K}^2+m^2). \tag{2.17}$$
Here the horizontal radial wave-number $\mathscr{K} = (k^2+l^2)^{\frac{1}{2}}$ has been introduced ($\mathscr{K} \geqslant 0$). A vorticity equation now follows from equations (2.11) and (2.17):
$$\ddot{G} + \Delta g\mathscr{K} G = -(a/\pi)\int_{-\infty}^{\infty} dm(i\Delta g m\dot{\xi} + \Delta\dddot{\xi})/(\mathscr{K}^2+m^2).$$
If here t is replaced by τ and the equation is multiplied by $\sin[(\Delta g\mathscr{K})^{\frac{1}{2}}(t-\tau)]$ and integrated from $\tau = t_0$ to $\tau = t$, $(t > t_0)$, a familiar process of integration by parts (twice) yields
$$\phi(x,l,z,t) = -(a/2\pi^2)\int_{-\infty}^{\infty} dk\, e^{ikx}\int_{-\infty}^{\infty} dm\Big[(e^{imz} + \Delta\, e^{-\mathscr{K}|z|}\operatorname{sgn} z)\,\dot{\xi}(l,m,t)$$
$$+ (\Delta g\mathscr{K})^{\frac{1}{2}}(im-\Delta)\,\mathscr{K}e^{-\mathscr{K}|z|}\operatorname{sgn} z\int_{t_0}^{t} d\tau\,\dot{\xi}(l,m,\tau)\sin\{(\Delta g\mathscr{K})^{\frac{1}{2}}(t-\tau)\}\Big]\Big/(\mathscr{K}^2+m^2), \tag{2.18}$$
where the initial conditions (2.13) have been used once more. The Green's function for the problem now readily follows:
$$\phi(\mathbf{r},t) = \int_{-\infty}^{\infty} dY\int_{-\infty}^{\infty} dZ\,\xi_0(Y,Z)\,G(\mathbf{r},t,Y,Z),$$
where
$$G(\mathbf{r},t,Y,Z) = (ia/4\pi^3)\int_{-\infty}^{\infty} dk\int_{-\infty}^{\infty} dl\int_{-\infty}^{\infty} dm\Big[(1+\Delta\exp(imz-\mathscr{K}|z|)\operatorname{sgn} z)$$
$$\times\,\mathbf{K}.\dot{\mathbf{r}}_0(t)\exp\{-i\mathbf{K}.\mathbf{r}_0(t)\} + (\Delta y/\mathscr{K})^{\frac{1}{2}}(im-\Delta\mathscr{K})\exp(-\mathscr{K}|z|)$$
$$\times\operatorname{sgn} z\int_{t}^{t_0} dt\sin\{(\Delta g\mathscr{K})^{\frac{1}{2}}(t-\tau)\}\,\mathbf{K}.\dot{\mathbf{r}}_0(\tau)\exp\{-i\mathbf{K}.\mathbf{r}_0(\tau)\}\Big](\mathscr{K}^2+m^2)^{-1}$$
$$\times\exp i\{kx + l(y-Y) + m(z-Z)\}. \tag{2.19}$$
Here the m-integration is readily performed but the result is an expression more cumbersome than the one given.

3. The drag and wave energy

Consider now the hydrodynamic presure. This is given approximately by $p = -\rho\dot{\phi}$, if $\alpha \ll 1$, and so the resultant hydrodynamic thrust on the disk at time t is approximately
$$\mathbf{F}(t) = -2a\int_{-\infty}^{\infty} dy\int_{-\infty}^{\infty} dz\,\rho\,\partial\phi(0,y,z,t)/\partial t[0, \partial\xi(y,z,t)/\partial y, \partial\xi(y,z,t)/\partial z] \tag{3.1}$$

if terms $O(\alpha^3)$ are neglected. In this expression

$$(1/4\pi^2 i)\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\mu[0,\lambda,\mu]\,\xi(-\lambda,\,-\mu,t)\exp\{-i(\lambda y+\mu z)\}$$

is substituted for $[0, \partial\xi/\partial y, \partial\xi/\partial z]$, and

$$\int_{-\infty}^{\infty} dy\, e^{-i\lambda y}\,\partial\phi(0,y,z,t)/\partial t$$

is identified as a Fourier transform of $\partial\phi/dt$. The expression for the thrust may then be written

$$\mathbf{F}(t) = (i\bar{\rho}a/2\pi^2)\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\mu[0,\lambda,\mu]\xi(-\lambda,\,-\mu,t) \times \int_{-\infty}^{\infty} dz\,\partial\phi(x=0,\,\lambda,z,t)/\partial t\, e^{-i\mu z}(1-\Delta\,\mathrm{sgn}\,z). \tag{3.2}$$

Here $\partial\phi/\partial t$ may be found from (2.18) by a differentiation with respect to the time and the substitutions $x = 0$ and $l = \lambda$. A further substitution for $\xi(\lambda,\, m,\, t)$ and $\xi(-\lambda,\,-\mu,t)$ from equation (2.16) then enables the thrust to be expressed as

$$\mathbf{F}(t) = \mathbf{F}_M(t)+\mathbf{F}_G(t), \tag{3.3}$$

where
$$\begin{aligned}\mathbf{F}_M(t) = (\bar{\rho}a^2/2\pi^4)&\int_{-\infty}^{\infty} dk\int_{-\infty}^{\infty} dl\int_{-\infty}^{\infty} dm\int_{-\infty}^{\infty} d\mu \\ &\times[0,l,\mu]\xi_0(l,m)\xi_0(-l,\,-\mu)\exp\{i(\mu-m)z_0(t)\}\,(\mathscr{K}^2+m^2)^{-1}\\ &\times\{\Delta(\mu-i\Delta\mathscr{K})\,(\mathscr{K}^2+\mu^2)^{-1}+\Delta(m-\mu)^{-1}+\pi i\delta(m-\mu)\}\\ &\times\{(l\dot{y}_0(t)+m\dot{z}_0(t))^2+i(l\ddot{y}_0(t)+m\ddot{z}_0(t))\}\end{aligned} \tag{3.4}$$

and
$$\begin{aligned}\mathbf{F}_G(k) = (\bar{\rho}\eta^2\Delta g/2\pi 4)&\int_{-\infty}^{\infty} dk\int_{-\infty}^{\infty} dl\int_{-\infty}^{\infty} dm\int_{-\infty}^{\infty} d\mu[0,l,\mu]\,\xi_0(l,m)\\ &\times\,\xi_0\{i(l y_0(t)(-l,\,-\mu)\,(\mathscr{K}^2+\mu^2)^{-1}(\mathscr{K}^2+m^2)^{-1}(i\mu+\Delta\mathscr{K})\,(im-\Delta\mathscr{K})\\ &\times\exp\{i(ly_0(t)+\mu z_0(t))\}\int_{t_0}^{t} d\tau(l\dot{y}_0(\tau)+m\dot{z}_0(\tau))\exp\{-i(ly_0(\tau)+mz_0(\tau))\}\\ &\times\cos\{(\Delta g\mathscr{K})^{\frac{1}{2}}\,(t-\tau)\}.\end{aligned} \tag{3.5}$$

In (3.4), the integral is a Cauchy principal value. Also the identity

$$\int_{-\infty}^{\infty} dz\exp\{-i\mu z-\mathscr{K}|z|\}\,(\Delta-\mathrm{sgn}\,z) = 2(i\mu+\Delta\mathscr{K})/(\mathscr{K}^2+\mu^2)$$

and others similar to this have been used.

Several checks may be made on these drag formulae. For instance, if $\Delta = 0$, $\mathbf{F}_G(t)$ vanishes and

$$\mathbf{F}_M(t) = -(\bar{\rho}a^2/2\pi^3)\int_{-\infty}^{\infty} dk\int_{-\infty}^{\infty} dl\int_{-\infty}^{\infty} dm[0,l,m]\,\xi_0(l,m) \times\,\xi_0(-l,\,-m)\,(l\ddot{y}_0(t)+m\ddot{z}_0(t))/(\mathscr{K}^2+m^2). \tag{3.6}$$

This is a formula for the added mass effect in a uniform fluid. Setting $\dot{y}_0(t) = 0$, from (3.5) there follows the drag formula of Warren (1968, equation (3.18)).

Next, suppose that the motion is horizontal, $\dot{z}_0(t) = 0$, and that the speed is constant, $\ddot{y}_0(t) = 0$ and $\dot{y}_0(t) = V$ say. Consider then the limit of $\mathbf{F}_G(t)$ as t becomes infinite and let T be the horizontal component of this limiting form of the drag. Then the τ and k integrations are readily performed. Setting $l = \Delta g V^{-2} \sec\theta$, one finds after some manipulation that

$$T = (2a^2\bar{\rho}\Delta^2 g^2/\pi V^2)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta |R|^2 \sec^3\theta, \tag{3.7}$$

where

$$R = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\, \partial\xi_0(y,z)/\partial y (1-\Delta\,\mathrm{sgn}\,z)\exp\{-\Delta g V^{-2}\sec\theta(|z|\sec\theta - iy)\}. \tag{3.8}$$

This is Michell's wave-resistance formula for the steady horizontal motion of a thin ship if Δ is set equal to one (Michell 1898).

The application of the above formulae to the case of ascent through the interface is straightforward. The work done against the drag during the time interval (t_0, t_1) is

$$-\int_{t_0}^{t_1} \mathbf{F}\,.\,\dot{\mathbf{r}}_0(t)\,dt$$

and so the work done in generating the waves is

$$-\int_{t_0}^{t_1} \mathbf{F}_G\,.\,\dot{\mathbf{r}}_0(t)\,dt.$$

Applying the formula (3.5), one finds that this is

$$\begin{aligned} &-(\bar{\rho}a^2\Delta g/2\pi^4)\int_{-\infty}^{\infty} dk\int_{-\infty}^{\infty} dl\int_{-\infty}^{\infty} dm\int_{-\infty}^{\infty} d\mu\,\xi_0(l,m) \\ &\quad\times \xi_0(-l,-\mu)\frac{(i\mu+\Delta\mathscr{K})(im-\Delta\mathscr{K})}{(\mathscr{K}^2+\mu^2)(\mathscr{K}^2+m^2)}\int_{t_0}^{t_1} dt\, d\{\exp i(ly_0(t)+\mu z_0(t))\}/dt \\ &\quad\times\int_{t_0}^{t_1} d\tau\, d\{\exp i(-ly_0(\tau)-mz_0(\tau))\}/d\tau\cos\{(\Delta g\mathscr{K})^{\frac{1}{2}}(t-\tau)\}. \end{aligned} \tag{3.9}$$

In the case of steady motion, $\dot{\mathbf{r}}_0(t) = (0, V, W)$ where V and W are constant, $W \neq 0$, and the time integrations can be made in terms of elementary functions. If then $t_1 \to \infty$ and $t_0 \to -\infty$, the resulting expression gives an estimate of the gravity wave energy generated during a transit of the interface. The identity

$$\begin{aligned} \lim_{t_1\to\infty}\int_{-t_1}^{t_1} dt\exp\{i(Vl+W\mu)t\}\int_{-t_1}^{t_1} d\tau\exp\{-i(Vl+Wm)\tau\}\cos\{(\Delta g\mathscr{K})^{\frac{1}{2}}(t-\tau)\} \\ = (\pi^2/W^2)[\delta\{\mu+W^{-1}(Vl-(\Delta g\mathscr{K})^{\frac{1}{2}})\}\delta\{m+W^{-1}(Vl-(\Delta g\mathscr{K})^{\frac{1}{2}})\} \\ +\delta\{\mu+W^{-1}(Vl+(\Delta g\mathscr{K})^{\frac{1}{2}})\}\delta\{m+W^{-1}(Vl+(\Delta g\mathscr{K})^{\frac{1}{2}})\}] \\ -2\pi i(Vl+Wm)\delta(m-\mu)/W\{\Delta g\mathscr{K}-(Vl+Wm)^2\} \end{aligned}$$

is required. Hence the total wave energy E_G is given by

$$\begin{aligned} E_G = (a^2\Delta^2\bar{\rho}g^2/2\pi^2W^2)\int_{-\infty}^{\infty} dk\int_{-\infty}^{\infty} dl(k^2+l^2)^{\frac{1}{2}}\{|\xi_0(l,m^*)|^2+|\xi_0(-l,m^*)|^2\} \\ \times\{m^{*2}+\Delta^2(k^2+l^2)\}/\{m^{2*}+k^2+l^2\}^2, \end{aligned} \tag{3.10}$$

where $$m^* = \{(\Delta_p \mathscr{K})^{\frac{1}{2}} + Vl\}/W. \tag{3.11}$$

This result may be compared with an earlier result of Warren (1961, equation (19)′): a change to polar co-ordinates $(k, l) \rightarrow (K, \theta)$ and the substitution

$$\xi_0(l, m) \rightarrow [(2 \sin Bl)/l]\, \xi_0(m)$$

leads to this earlier result if one sets $V = 0$ and considers the limiting process: $\lim_{B\rightarrow\infty} (E_G/2B)$. This gives the wave energy per unit length of an ascending body for the two-dimensional case. The axi-symmetric case is difficult to compare because it is not clear that the boundary conditions at the hull of the thin disk may be adapted consistently. However, the *form* of integral for the axi-symmetric case (Warren 1961, equation (19)) is readily seen to be the same as that for the thin disk if in this latter case one considers the limit in which

$$\xi_0(l, m) \rightarrow [\lim_{B\rightarrow 0} (2 \sin Bl/Bl)]\, \xi_0(m).$$

However the multiplying factors outside the corresponding integrals do not agree. This modification is hardly surprising since the boundary conditions for the thin disk are somewhat different from those which are applied to the axi-symmetric case.

4. Calculations for a specified case

A numerical evaluation of the formula (3.10) was carried out for a disk whose thickness is given in the form

$$\xi_{00}(Y', Z') = A^2B^2/(A^2 + Y'^2)(B^2 + Z'^2), \tag{4.1}$$

where ξ_{00} represents the form relative to the major and minor axes of symmetry Y' and Z'. If β is the inclination angle between the major axis and the vertical, then in terms of wave-numbers,

$$\xi_0(l, m) = \xi_{00}(l \sin\beta + m \cos\beta, m \sin\beta - l \cos\beta). \tag{4.2}$$

The aspect ratio B/A selected was 0·1. The contours of the disk then approximate to ellipses of high eccentricity, except for a slight broadening in the vicinity of the minor axis. The effect of this increase in girth 'amidships' appears to be exaggerated in the curves of the wave drag and suggests that at certain angles of incidence γ and inclination β small protuberances can make important contributions to the drag. The curves are shown in figures 2–5, where E_0 is the dimensionless form of the wave energy E_G:

$$E_G = (2\pi^2 \bar{\rho} \mathbf{U}^4 A_0^2 B_0^2 a^2/g\Delta) E_0(A_0; \beta, \gamma), \tag{4.3}$$

where the Froude numbers $A_0 = \Delta g A/\mathbf{U}^2$, $B_0 = \Delta g B/\mathbf{U}^2$ have been introduced, and $\mathbf{U}^2 = V^2 + W^2$. However, in some cases the values of $A_0^{-1} E_0$ are given so as to simplify the plottings. Also because of restrictions of computer time, the case in which Δ is small and g is large has been considered only. Thus in (3.10) one sets $\Delta g = g'(+0)$ and $\Delta = 0$, so that $A = g'A/\mathbf{U}^2$.

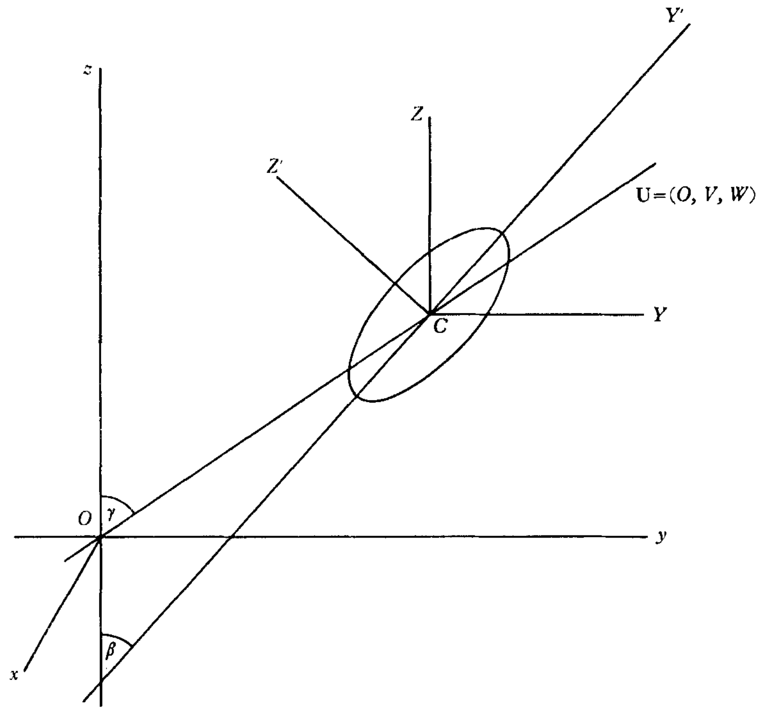


FIGURE 1. Axes of reference and configuration of disk.

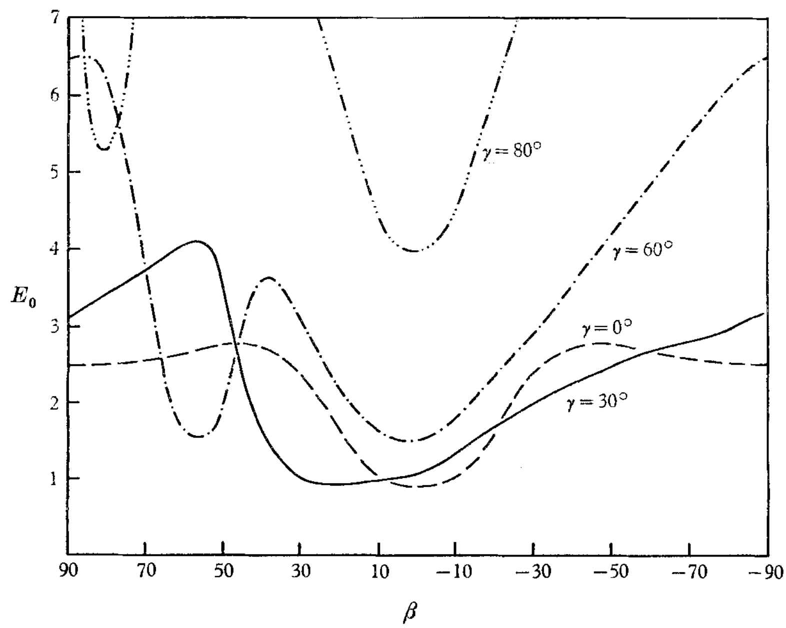


FIGURE 2. Curves of energy E versus angle of inclination of major axis of symmetry, β, for various angles of incidence γ. Froude number $A_0 = \Delta gA/\mathrm{U}^2 = 1$.

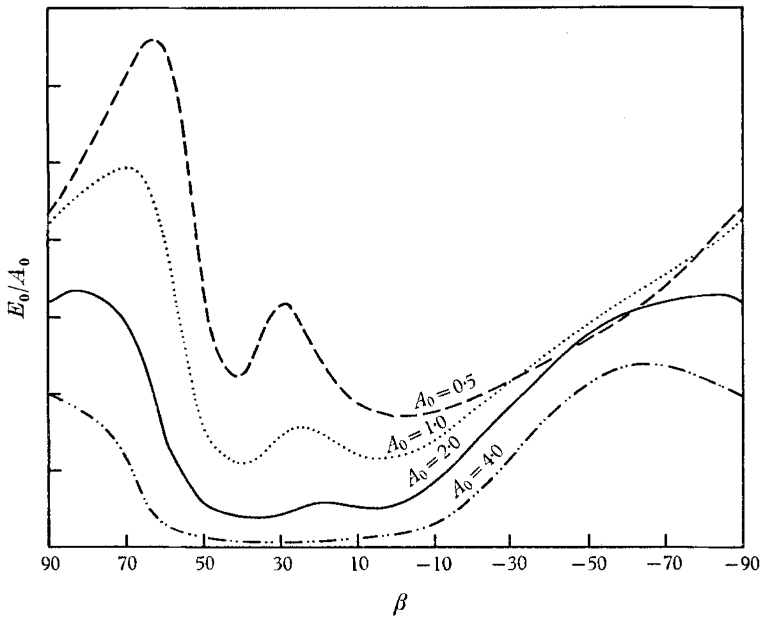


FIGURE 3. Energy versus angle of inclination for various Froude numbers for an angle of incidence $\gamma = 45$. Froude number $A_0 = \Delta gA/U^2$.

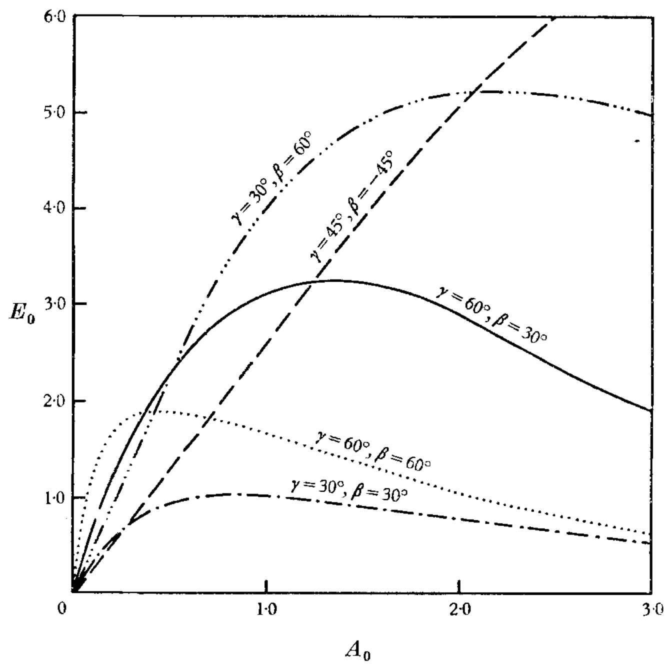
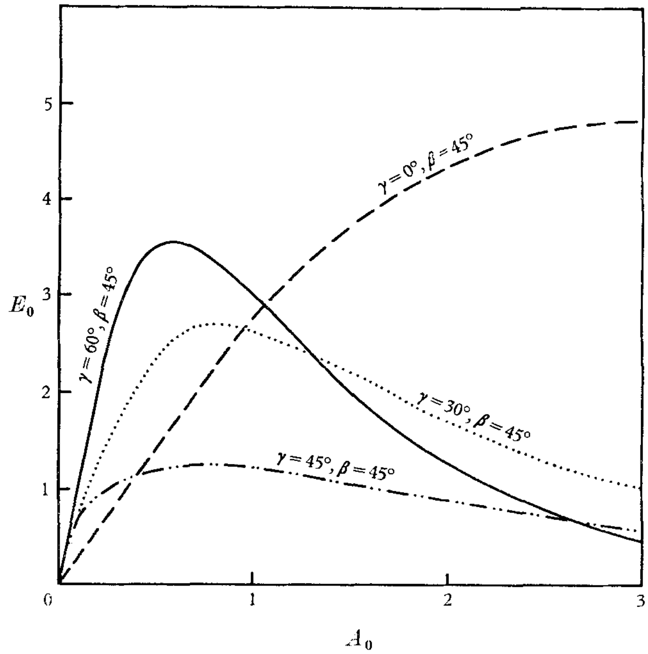


FIGURE 4. For legend see figure 5.

The numerical computation entailed the use of certain specially devised techniques, because the evaluation of the formula is not so straightforward as it might at first appear. Tables of weights and abscissae for Gauss–Laguerre quadratures are given by Rabinowitz & Weiss (1959), and it may be seen that



FIGURES 4 AND 5. Energy versus Froude number for various angles of incidence and inclination.

this method is inadequate for certain values of γ and β. For example, when $\gamma = 80$, $\beta = 0$, small wavelengths contribute significantly to the energy. Thus a special technique was devised based on Romberg's method of quadrature, which, however, was time consuming and required elaborate programming. For further details on this see MacKinnon (1968). The computation was carried out on the London University Computer Atlas.

Thanks are due to the Canadian Defence Research Board for financial support to one of us during this work, and to the referees for their valuable suggestions.

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